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REGULARIZATION OF NECESSARY CONDITIONS FOR A NONLOCAL BOUNDARY VALUE PROBLEM FOR A THREE-DIMENSIONAL MIXED-TYPE EQUATION

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Abstract. The presented work is devoted to the study of the solution of a boundary value problem for a three-dimensional mixed-type equation with nonlocal boundary conditions in a bounded domain with a Lyapunov boundary. In the upper half-space, this is an elliptic-type equation, or the Laplace equation, and in the lower half-space, it is a hyperbolic-type equation. Necessary solvability conditions are derived in both the elliptic and hyperbolic cases. Regularization of singular necessary conditions is carried out using a special scheme. The regularized relations obtained in this work are a tool for proving the Fredholm property of the problem.

Keywords: Differential equation of mixed type, three-dimensional equation of hyperbolic type, three-dimensional equation of elliptic type, nonlocal boundary conditions, basic relations, fundamental solution, regular necessary conditions, singular necessary conditions, regularization.

AMS Subject Classification: 35M12.

1. Introduction

One of the intensively developing sections of the modern theory of partial differential equations, due to its theoretical and applied importance, is the theory of boundary value problems for equations of mixed type. The beginning of the study of boundary value problems for equations of mixed type was laid in the works of F.O. Tricomi and S. Gellerstedt [14],[8]. The next stage in the development of the theory of boundary value problems for equations of mixed type were the works of M.A. Lavrentyev, A.V. Bitsadze [10],[6], K.I. Babenko [2], F.I. Frankl [7], M.E. Lerner [11]. The applied importance of the theory of boundary value problems for equations of mixed type lies in the fact that such problems find their application in the momentless theory of shells with alternating curvature, problems of transonic and supersonic gas mechanics, the theory of infinitesimal bending of surfaces, magnetohydrodynamic flows with a period through critical velocities, etc.

In this paper, we consider a nonlocal boundary value problem for a three-dimensional mixed-type equation in second-order partial derivatives, which turns out to be an elliptic-type equation in the subsonic region and a hyperbolic-type equation in the supersonic region.

In contrast to classical problems, we studied equations of both even and odd orders and investigated the Fredholm property of many three-dimensional problems with non-local boundary conditions for both typical and non-typical differential equations [9]. Non-local boundary conditions are such that the entire boundary is a support for each boundary condition. The number of non-local boundary conditions can be taken equal to the order of the equation, which allows us to eliminate the misunderstanding between equations of even and odd orders.

The tool for proving the Fredholm property of the problem is the regularization of the necessary solvability conditions [1],[12],[13],[16].

The idea of necessary conditions for partial differential equations was first used by A.V. Bitsadze for the Laplace equation [5, p.185] in both two-dimensional and three-dimensional cases. But the regularization of singularities in the necessary conditions was artificial, especially in three-dimensional cases, and contained some uncertainties.

Finally, Beger derived these necessary conditions for the Cauchy-Riemann equation [3],[4].

A part of the necessary conditions obtained for the posed three-dimensional problem contains singular integrals. But the regularization of these singularities does not follow the generally accepted scheme.

As is known, the regularization of singular integral equations in the classical case is carried out by the method of successive substitutions: after the first substitution, a double singular integral is obtained, and when changing the order of integration in the double integral, the Poincaré-Bertrand formula is used to obtain a regular kernel and a jump that does not "eat up" the external function. Thus, in this case we obtain a Fredholm integral equation of the second kind with a regular kernel.

In our case, the obtained necessary solvability conditions, or integral equations, are in the spectrum, therefore, when they are regularized according to the indicated scheme, we arrive at Fredholm integral equations of the first kind, which is a "dead end".

According to the proposed unique scheme, singular necessary conditions are regularized using specified boundary conditions, which is fundamentally new [1],[12],[13],[16]. The obtained regularized relations allow us to further prove the Fredholm property of the problem.

2. Problem statement.

Let us consider an equation of mixed type

$$sign\, x_3 \frac{\partial^2 u(x)}{\partial x_3^2} + \left(\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} \right) = 0 \tag{1}$$

in a three-dimensional bounded domain $D=\{x=(x_1,x_2,x_3)\in R^3\}$ with a Lyapunov boundary Γ, convex in the direction Ox_3, with nonlocal boundary conditions:

$$l_i u=\sum_{j=1}^{3}\left[\alpha_{ij1}^{(1)}(x')\frac{\partial u_1(x)}{\partial x_j}\Big|_{x_3=\gamma_1(x')}+\alpha_{ij1}^{(0)}(x')\frac{\partial u_1(x)}{\partial x_j}\Big|_{x_3=\gamma_0(x')}+\right.$$

$$\left.+\alpha_{ij2}^{(2)}(x')\frac{\partial u_2(x)}{\partial x_j}\Big|_{x_3=\gamma_2(x')}+\alpha_{ij2}^{(0)}(x')\frac{\partial u_2(x)}{\partial x_j}\Big|_{x_3=\gamma_0(x')}\right]+$$

$$+\alpha_{i1}^{(1)}(x')u_1(x',\gamma_1(x'))+\alpha_{i1}^{(0)}(x')u_1(x',\gamma_0(x'))+$$

$$+\alpha_{i2}^{(2)}(x')u_2(x',\gamma_2(x'))+\alpha_{i2}^{(0)}(x')u_2(x',\gamma_0(x'))=f_i(x'), \tag{2}$$

$$i=1,2;\ \ x'=(x_1,x_2)\in S,$$

$$u(x)=\begin{cases}u_1(x),\ x\in D_1=\{x\in D, x_3>0\},\\ u_2(x),\ x\in D_2=\{x\in D\, x_3<0\},\end{cases}$$

$$u(x)=f_0(x),\ \ x\in L=\overline{\Gamma}_1\cap\overline{\Gamma}_2, \tag{3}$$

where domain $S\subset Ox_1x_2$ is the projection of domain D onto plane $Ox_1x_2=Ox'$, Γ_1 and Γ_2 are the lower and upper halfsurfaces of boundary Γ respectively, defined as follows: $\Gamma_k=\{\xi=(\xi_1,\xi_2,\xi_3):\xi_3=\gamma_k(\xi'),\ \xi'=(\xi_1,\xi_2)\in S\}$, where $\xi_3=\gamma_k(\xi_1,\xi_2), k=1,2$, are halfsurfaces Γ_1 and Γ_2 respectively; functions $\gamma_k(\xi'), k=1,2$, are twice differentiable with respect to ξ_1,ξ_2 in the domain S; L is the equator, connecting halfsurfaces Γ_1 and Γ_2: $L=\overline{\Gamma}_1\cap\overline{\Gamma}_2$.

Let us designate the projection of the domain D onto the plane $Ox_1x_2=Ox'$ as $\Gamma_0=\{\xi=(\xi_1,\xi_2,\xi_3):\xi_3=\gamma_0(\xi')=0,\ \xi'\in S\}$. The coefficients $\alpha_{ijp}^{(k)}(x'),\alpha_{ip}^{(k)}(x')$, $i,p=1,2;$ $j=1,2,3;$ $k=0,1$, are continuous in S and $\alpha_{i2}^{(k)}(\xi')$, $k=0,2$, $\alpha_{ij1}^{(k)}(\xi')$, $k=0,1$, $j=\overline{1,3}$, $\alpha_{i32}^{(0)}(\xi')$, $\alpha_{ij2}^{(2)}(\xi')$, $i=1,2; j=\overline{1,3}, \xi'\in S$, satisfy the Hölder condition in the domain S, the right-hand sides $f_i(x')\in C^1(S),\ f_i(x')\big|_{\partial S}=0,\ i=1,2$, and $f_0(x)$ is continuous on L.

Linearly independent boundary conditions (2) as if "sew" the values of the desired function and its partial derivatives on the upper and lower half-surfaces Γ_1, Γ_2 and Γ_0.

So, for $x_3 > 0$ we have an equation of elliptic type

$$\Delta_3 u_1(x) = 0,\ x \in D_1, \tag{4}$$

at that $\Gamma' = \Gamma_0 \cup \Gamma_1$ is ∂D_1, and for $x_3 < 0$ we have a hyperbolic type equation

$$\frac{\partial^2 u_2(x)}{\partial x_3^2} = \frac{\partial^2 u_2(x)}{\partial x_1^2} + \frac{\partial^2 u_2(x)}{\partial x_2^2},\ x \in D_2, \tag{5}$$

and the boundary $\Gamma'' = \Gamma_0 \cup \Gamma_2$ is ∂D_2.

Remark 1. The boundary condition (3) is given on a zero measure set and does not violate the generality of the solution to problem (1)-(2) but later helps to prove the Fredholm property of the stated problem.

3.The equation of elliptic type.

Now we are going to consider equation (4)

$$Lu_1 = \Delta u_1(x) = \frac{\partial^2 u_1(x)}{\partial x_1^2} + \frac{\partial^2 u_1(x)}{\partial x_2^2} + \frac{\partial^2 u_1(x)}{\partial x_3^2} = 0, \quad x \in D_1, \tag{6}$$

with boundary conditions (2).

The fundamental solution for the three-dimensional Laplace equation has the form [15]:

$$U_1(x-\xi) = -\frac{1}{4\pi}\frac{1}{|x-\xi|}. \tag{7}$$

Let us get the basic relationships and necessary conditions in elliptic case.

Multiplying equation (6) by the fundamental solution (7), integrating it over the domain D_1 and taking into account that $\Delta_x U_1(x-\xi) = \delta(x-\xi)$ where $\delta(x-\xi)$ is the Dirac δ -function we'll get **the first basic relationship**:

$$-\sum_{j=1}^{3}\int_{\Gamma'}\left[\left(\frac{\partial u_1(x)}{\partial x_j}U_1(x-\xi) - u_1(x)\frac{\partial U_1(x-\xi}{\partial x_j}\right)\cos(\nu, x_j)dx\right] =$$

$$= \int_{D_1} u_1(x)\delta(x-\xi)dx = \begin{cases} u_1(\xi), & \xi \in D_1, \\ \frac{1}{2}u_1(\xi), & \xi \in \Gamma'. \end{cases} \tag{8}$$

Here the first relationship gives the representation of the general solution of equation (6) and the second expression in (8) is the **first necessary condition.**

As

$$\frac{\partial U_1(x-\xi)}{\partial x_i} = \frac{x_i - \xi_i}{4\pi|x-\xi|^3} = \frac{\cos(x-\xi, x_i)}{4\pi|x-\xi|^2} \tag{9}$$

then the first necessary condition takes the form:

$$\frac{1}{2}u_1(\xi) = -\int_{\Gamma'} \frac{\partial u_1(x)}{\partial \nu} U_1(x-\xi)dx + \int_{\Gamma'} u_1(x) \sum_{j=1}^{3} \left(\frac{\cos(x-\xi, x_j)}{4\pi |x-\xi|^2} \right) \cos(\nu_x, x_j) dx ,$$

$$\xi \in \Gamma', \quad (10)$$

where all the integrands have a weak singularity as the order of singularity doesn't exceed the multiplicity of the integrals.

As $\Gamma' = \Gamma_0 \cup \Gamma_1$ then for $\xi' \in \Gamma_k, k = 0,1$, **the first necessary conditions** (10) take the form:

$$u_1(\xi', \gamma_k(\xi')) = \frac{(-1)^k}{2\pi} \int_S \left(\frac{u_1(x)}{|x-\xi|^2} \cos(\nu_x, x-\xi) \right) \Bigg|_{\substack{x_3=\gamma_k(x') \\ \xi_3=\gamma_k(\xi')}} \frac{dx'}{\cos(\nu_x, x_3)} -$$

$$-\frac{(-1)^k}{2\pi} \int_S \left(\frac{\partial u_1(x)}{\partial \nu} \frac{1}{|x-\xi|^2} \right) \Bigg|_{\substack{x_3=\gamma_k(x') \\ \xi_3=\gamma_k(\xi')}} \frac{dx'}{\cos(\nu_x, x_3)}, k = 0,1 . \quad (11)$$

Thus, we have proved

Theorem 1. *Let a convex along the direction* Ox_3 *domain* $D \subset R^3$ *be bounded with the boundary* Γ *which is a Lyapunov surface. Then the obtained first necessary condition* (11) *is regular*.

Multiplying (6) by $\frac{\partial U_1(x-\xi)}{\partial x_i}, i = \overline{1,3},$ and integrating it over the domain D_1 we obtain **the second basic relationships**:

$$\int_{\Gamma} \frac{\partial u_1(x)}{\partial x_i} \frac{\partial U_1(x-\xi)}{\partial \nu_x} dx +$$

$$+\int_{\Gamma} \frac{\partial u_1(x)}{\partial x_m} \left[\frac{\partial U_1(x-\xi)}{\partial x_i} \cos(\nu_x, x_m) - \frac{\partial U_1(x-\xi)}{\partial x_m} \cos(\nu_x, x_i) \right] dx +$$

$$+\int_{\Gamma} \frac{\partial u_1(x)}{\partial x_l} \left[\frac{\partial U_1(x-\xi)}{\partial x_i} \cos(\nu_x, x_l) - \frac{\partial U_1(x-\xi)}{\partial x_l} \cos(\nu_x, x_i) \right] dx =$$

$$= \begin{cases} \dfrac{\partial u_1(\xi)}{\partial \xi_i}, & \xi \in D_1, \\ \dfrac{1}{2} \dfrac{\partial u_1(\xi)}{\partial \xi_i}, & \xi \in \Gamma', \end{cases} \quad i = \overline{1,3}, \quad (12)$$

where the numbers *i*, *m*, *l* make a permutation of numbers 1,2,3.

The second expressions in (12) are **the second necessary conditions** **(** $\xi \in \Gamma', i = \overline{1,3}$):

$$\frac{1}{2}\frac{\partial u_1(\xi)}{\partial \xi_i} =$$

$$= \int_\Gamma \frac{\partial u_1(x)}{\partial x_m}\left[\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu_x, x_m) - \frac{\partial U_1(x-\xi)}{\partial x_m}\cos(\nu_x, x_i)\right]dx +$$

$$+\int_\Gamma \frac{\partial u_1(x)}{\partial x_l}\left[\frac{\partial U_1(x-\xi)}{\partial x_i}\cos(\nu_x, x_l) - \frac{\partial U_1(x-\xi)}{\partial x_l}\cos(\nu_x, x_i)\right]dx +$$

$$+\int_\Gamma \frac{\partial u_1(x)}{\partial x_i}\frac{\partial U_1(x-\xi)}{\partial \nu_x}dx,\ \ \xi \in \Gamma, i = \overline{1,3}\ , \tag{13}$$

where the numbers i,m,l make a permutation of numbers 1,2,3.

In the virtue of (9) and introducing the designations

$$K_{ij}(x,\xi) = \left(\cos(x-\xi, x_i)\cos(\nu_x, x_j) - \cos(x-\xi, x_j)\cos(\nu_x, x_i)\right) \tag{14}$$

we can rewrite the necessary conditions (14) in the form:

$$\frac{1}{2}\frac{\partial u_1(\xi)}{\partial \xi_i} = \int_{\Gamma'} \frac{\partial u_1(x)}{\partial x_m}\frac{K_{im}(x,\xi)}{4\pi|x-\xi|^2}dx + \int_{\Gamma'} \frac{\partial u_1(x)}{\partial x_l}\frac{K_{il}(x,\xi)}{4\pi|x-\xi|^2}dx$$

$$+\int_{\Gamma'} \frac{\partial u_1(x)}{\partial x_i}\frac{\partial U_1(x-\xi)}{\partial \nu_x}dx \tag{15}$$

where the numbers i,m,l make a permutation of numbers 1,2,3.

Taking into account that $\Gamma' = \Gamma_1 \cup \Gamma_0$ we introduce the second group of necessary conditions by disclosing two first surface integrals in relationship (15) ($i = \overline{1,3}$) over the upper half surface Γ_1 and Γ_0, extracting only singular terms:

$$\frac{1}{2}\frac{\partial u_1}{\partial \xi_i}\Big|_{\xi_3=\gamma_k(\xi')} = \int_S \frac{\partial u_1(x)}{\partial x_m}\Big|_{x_3=\gamma_k(x')}\frac{K_{im}(x,\xi)}{4\pi|x-\xi|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x, x_3)} +$$

$$+\int_S \frac{\partial u_1(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')}\frac{K_{il}(x,\xi)}{4\pi|x-\xi|^2}\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x, x_3)} + \ldots, k = 0,1, \tag{16}$$

where three dots designate the sum of nonsingular terms.

Let us introduce the designations:

$$K_{ij}^{(m)}(x',\xi') == K_{ij}(x,\xi)\Bigg|_{\substack{x_3=\gamma_m(x')\\ \xi_3=\gamma_m(\xi')}}, m = 0,1. \tag{17}$$

By Lagrange formula:

$$\left|x-\xi\right|^2\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}=\left|x'-\xi'\right|^2\ P_k(x',\xi'),k=0,1, \tag{18}$$

where

$$P_k(x',\xi')=1+\sum_{m=1}^{2}\left(\frac{\partial\gamma_k(x')}{\partial x_m}\right)^2\cos^2(x'-\xi',x_m)+O(\left|x'-\xi'\right|).$$

R e m a r k 2. Notice that for $\xi'=x'$ we have:

$$P_k(x',x')=1+\left(\frac{\partial\gamma_k}{\partial x_1}+\frac{\partial\gamma_k}{\partial x_2}\right)^2\neq 0,k=0,1.$$

By means of the designations (17), (18) we'll rewrite **the first necessary conditions** (11) for k=0,1 as follows:

$$u_1(\xi',\gamma_k(\xi'))=(-1)^k\frac{1}{2\pi}\int_S\frac{u_1(x)}{P_k(x',\xi')\left|x'-\xi'\right|^2}\cos(\nu_x,x-\xi)\Bigg|_{\substack{x_3=\gamma_k(x')\\ \xi_3=\gamma_k(\xi')}}\frac{dx'}{\cos(\nu_x,x_3)}+...,$$

$$\xi'\in S,k=0,1, \tag{19}$$

and the necessary conditions (16) take the form:

$$\frac{1}{2}\frac{\partial u_1}{\partial\xi_i}\Big|_{\xi_3=\gamma_k(\xi')}=\int_S\frac{\partial u_1(x)}{\partial x_m}\Bigg|_{x_3=\gamma_k(x')}\frac{1}{4\pi\left|x'-\xi'\right|^2}\frac{K_{im}^{(k)}(x',\xi')}{P_k(x',\xi')}\frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+\int_S\frac{\partial u_1(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')}\frac{1}{4\pi\left|x'-\xi\right|'^2}\frac{K_{il}^{(k)}(x',\xi')}{P_k(x',\xi')}\frac{dx'}{\cos(\nu_x,x_3)}+...,\xi'\in S,i=\overline{1,3},k=0,1, \tag{20}$$

where i,m,l make permutation of numbers 1,2,3.

If we extract by Tailor series the term in the integrand of (20) which depends only on x' we'll obtain **the second necessary conditions** in the form:

$$\frac{1}{2}\frac{\partial u_1}{\partial\xi_i}\Big|_{\xi_3=\gamma_k(\xi')}=\int_S\frac{\partial u_1(x)}{\partial x_m}\Bigg|_{x_3=\gamma_k(x')}\frac{1}{4\pi\left|x'-\xi'\right|^2}\frac{K_{im}^{(k)}(x',x')}{P_k(x',x')}\frac{dx'}{\cos(\nu_x,x_3)}+$$

$$+\int_S\frac{\partial u_1(x)}{\partial x_l}\Big|_{x_3=\gamma_k(x')}\frac{1}{4\pi\left|x'-\xi'\right|^2}\frac{K_{il}^{(k)}(x',x')}{P_k(x',x')}\frac{dx'}{\cos(\nu_x,x_3)}+...,\xi'\in S,i=\overline{1,3},k=0,1. \tag{21}$$

Theorem 2. *Under assumptions of Theorem* 3.1 *the second necessary conditions* (21) *are singular*.

4. Equation of hyperbolic type.

Let us consider equation (5)

$$lu_2(x) = \frac{\partial^2 u_2(x)}{\partial x_3^2} - \left(\frac{\partial^2 u_2(x)}{\partial x_1^2} + \frac{\partial^2 u_2(x)}{\partial x_2^2} \right) = 0 \qquad (22)$$

in 3-dimensional domain $D_2 = \{x = (x_1, x_2, x_3), x_3 < 0\} \subset R^3$ with boundary $\Gamma'' = \Gamma_2 \cup \Gamma_0$.

The fundamental solution of (22) is [17]

$$U_2(x-\xi) = U_2(x_3-\xi_3, x_1-\xi_1, x_2-\xi_2) = = \frac{\theta((x_3-\xi_3) - \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2})}{2\pi\sqrt{(x_3-\xi_3)^2 - (x_1-\xi_1)^2 - (x_2-\xi_2)^2}}. \qquad (23)$$

Multiplying (22) by (23), integrating over D_2, applying Gauss-Ostrogradsky formula and taking into account that $l_x U_2(x-\xi) = \delta(x-\xi)$ [17] we obtain **the first basic relationship** for equation (22):

$$\int_{\Gamma''} \left[u_2(x) \frac{\partial U_2(x-\xi)}{\partial x_3} - \frac{\partial u_2(x)}{\partial x_3} U_2(x-\xi) \right] \cos(\nu_x, x_3) dx -$$

$$- \sum_{j=1}^{2} \int_{\Gamma''} \left[u_2(x) \frac{\partial U_2(x-\xi)}{\partial x_j} - \frac{\partial u_2(x)}{\partial x_j} U_2(x-\xi) \right] \cos(\nu_x, x_j) dx = \begin{cases} u_2(\xi), & \xi \in D_2, \\ \frac{1}{2} u_2(\xi), & \xi \in \Gamma''. \end{cases} \qquad (24)$$

The second of relationships (24) is called **the first necessary condition** of solvability of the stated problem:

$$\frac{1}{2} u_2(\xi) = \int_{\Gamma''} \left(u_2(x) \frac{\partial U_2(x-\xi)}{\partial x_3} - U_2(x-\xi) \frac{\partial u_2}{\partial x_3} \right) \cos(\nu_x, x_3) dx -$$
$$- \sum_{j=1}^{2} \int_{\Gamma''} \left(u_2(x) \frac{\partial U_2(x-\xi)}{\partial x_j} - U_2(x-\xi) \frac{\partial u_2(x)}{\partial x_j} \right) \cos(\nu_x, x_j) dx, \ \xi \in \Gamma''. \qquad (25)$$

Thus, we have established the following

Theorem 3. *Let the domain* $D_2 \subset R^3$ *with Lyapunov boundary* Γ'' *be bounded and convex in the direction of* Ox_3 *axis. Then the first basic relationship* (24) *for equation* (5), *or* (22) *holds true.*

Multiplying (22), or (5), by $\frac{\partial U_2(x-\xi)}{\partial x_i}$, $i=\overline{1,3}$, integrating over D_2 and then integrating by parts we obtain **the second basic relationships**:

$$\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_3}\left(\frac{\partial U_2(x-\xi)}{\partial x_i}\cos(\nu_x,x_3)-\frac{\partial U_2(x-\xi)}{\partial x_3}\cos(\nu_x,x_i)\right)dx+$$

$$+\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_i}\frac{\partial U_2(x-\xi)}{\partial x_3}\cos(\nu_x,x_3)dx-$$

$$-\sum_{j=1}^{2}\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_j}\left(\frac{\partial U_2(x-\xi)}{\partial x_i}\cos(\nu_x,x_j)-\frac{\partial U_2(x-\xi)}{\partial x_j}\cos(\nu_x,x_i)\right)dx+$$

$$+\sum_{j=1}^{2}\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_i}\frac{\partial U_2(x-\xi)}{\partial x_j}\cos(\nu_x,x_j)dx, \quad =\begin{cases}\frac{\partial u_2(\xi)}{\partial \xi_i},\ \xi\in D_2,\\ \frac{1}{2}\frac{\partial u_2(\xi)}{\partial \xi_i},\ \xi\in\Gamma'',\end{cases} \quad i=\overline{1,3}. \tag{26}$$

The second of the relations (26) is called **the second necessary condition** of solvability:

$$\frac{1}{2}\frac{\partial u_2(\xi)}{\partial \xi_i}=$$

$$=\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_3}\left(\frac{\partial U_2(x-\xi)}{\partial x_i}\cos(\nu_x,x_3)-\frac{\partial U_2(x-\xi)}{\partial x_3}\cos(\nu_x,x_i)\right)dx+$$

$$+\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_i}\frac{\partial U_2(x-\xi)}{\partial x_3}\cos(\nu_x,x_3)dx-$$

$$-\sum_{j=1}^{2}\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_j}\left(\frac{\partial U_2(x-\xi)}{\partial x_i}\cos(\nu_x,x_j)-\frac{\partial U_2(x-\xi)}{\partial x_j}\cos(\nu_x,x_i)\right)dx+$$

$$+\sum_{j=1}^{2}\int_{\Gamma''} \frac{\partial u_2(x)}{\partial x_i}\frac{\partial U_2(x-\xi)}{\partial x_j}\cos(\nu_x,x_j)dx,\ \xi\in\Gamma'',\ i=\overline{1,3}. \tag{27}$$

So, we have obtained the following

Theorem 4. *Under the conditions of Theorem* 3 *the second basic relations* (27) *are satisfied for solving equation* (22)**.**

It is easy to see that

$$\frac{\partial U_2(x-\xi)}{\partial x_3}=$$

$$\frac{1}{2\pi}\frac{\delta((x_3-\xi_3)-|x'-\xi'|)}{\sqrt{(x_3-\xi_3)^2-|x'-\xi'|^2}}-\frac{(x_3-\xi_3)\theta((x_3-\xi_3)-|x'-\xi'|)}{2\pi\sqrt{\left((x_3-\xi_3)^2-|x'-\xi'|^2\right)^3}}. \quad (28)$$

Similarly we obtain $\frac{\partial U_2(x-\xi)}{\partial x_i}$, $i=1,2$:

$$\frac{\partial U_2(x-\xi)}{\partial x_i}=$$

$$-\frac{1}{2\pi}\frac{(x_i-\xi_i)\delta((x_3-\xi_3)-|x'-\xi'|)}{|x'-\xi'|\sqrt{(x_3-\xi_3)^2-|x'-\xi'|^2}}+\frac{(x_i-\xi_i)\theta((x_3-\xi_3)-|x'-\xi'|)}{2\pi\sqrt{\left((x_3-\xi_3)^2-|x'-\xi'|^2\right)^3}},\ i=1,2. \quad (29)$$

As for $x_3=\gamma_2(x'),\ \xi_3=\gamma_2(\xi')$ we obtain that

$$x_3-\xi_3=\gamma_2(x')-\gamma_2(\xi')=\frac{\partial\gamma_2(x')}{\partial x_1}(x_1-\xi_1)+\frac{\partial\gamma_2(x')}{\partial x_2}(x_2-\xi_2)+o(|x'-\xi'|)$$

,

then introducing the notation

$$K_2(x',\xi')=\sqrt{\sum_{m=1}^{2}\left(\frac{\partial\gamma_2(x')}{\partial x_m}\right)^2\cos^2(x'-\xi',x_m)+O(|x'-\xi'|)} \quad (30)$$

we get that

$$\left(\gamma_2(x')-\gamma_2(\xi')\right)=|x'-\xi'|\ K_2(x',\xi'). \quad (31)$$

Remark 3. Let us notice that for $\xi'=x'$ we have:

$$K_2(x',x')=\sqrt{\left(\frac{\partial\gamma_2}{\partial x_1}\right)^2+\left(\frac{\partial\gamma_2}{\partial x_2}\right)^2+2\frac{\partial\gamma_2}{\partial x_1}\frac{\partial\gamma_2}{\partial x_2}}\neq 0.$$

Substituting (23), (28) and (29) into the necessary conditions (25) and (27) and adopting the notations (30) and (31), we rewrite the necessary conditions, selecting **only the singular terms** (the order of multiplicity of which is equal to or higher than the multiplicity of the integrals):

$$\frac{1}{2}u_2(\xi',0)=\sum_{j=1}^{2}\int_S u_2(x',0\left\{-\frac{(x_j-\xi_j)\theta((-|x'-\xi'|)}{2\pi i|x'-\xi'|^3}\right\}dx'+\ldots, \quad (32)$$

and for the case $\xi\in\Gamma_2$, $x\in\Gamma_2$:

$$\frac{1}{2}u_2(\xi',\gamma_2(x')) = \sum_{j=1}^{2}\int_S u_2(x',\gamma_2(x'))$$

$$\left\{-\frac{(x_j-\xi_j)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^3\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\right\}+\dots, \qquad (33)$$

as well for $i=1,2$:

$$\frac{1}{2}\frac{\partial u_2(\xi)}{\partial \xi_i}\Big|_{\xi_3=0} =$$

$$=\int_S \frac{\partial u_2(x)}{\partial x_3}\Big|_{x_3=0}\left\{-\frac{(x_i-\xi_i)\theta((-|x'-\xi'|)}{2\pi i|x'-\xi'|^3}\right\}\cos(\nu_x,x_3)dx'+$$

$$-\sum_{j=1}^{2}\int_{\Gamma''}\frac{\partial u_2(x)}{\partial x_j}\left(\left[\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|^3}\right]\cos(\nu_x,x_i)-\left[\frac{(x_i-\xi_i)\theta((-|x'-\xi'|)}{2\pi i|x'-\xi'|^3}\right]\cos(\nu_x,x_j)\right)$$

$$-\sum_{j=1}^{2}\int_{\Gamma''}\frac{\partial u_2(x)}{\partial x_i}\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|^3}\cos(\nu_x,x_j)dx+\dots\ . \qquad (34)$$

In the case $\xi\in\Gamma_2$, $x\in\Gamma_2$:

$$\frac{1}{2}\frac{\partial u_2(\xi)}{\partial \xi_i}\Big|_{\xi_3=\gamma_2(\xi')} =$$

$$\int_S \frac{\partial u_2(x)}{\partial x_3}\Big|_{x_3=\gamma_2(x')}\left\{-\frac{\left(K_2(x',\xi')-1\right)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^2\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\right\}\frac{\cos(\nu_x,x_i)dx'}{\cos(\nu_x,x_3)}-$$

$$-\int_S \frac{\partial u_2(x)}{\partial x_3}\Big|_{x_3=\gamma_2(x')}\left\{\frac{(x_i-\xi_i)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^3\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\right\}dx'+$$

$$+\int_S \frac{\partial u_2(x)}{\partial x_i}\Big|_{x_3=\gamma_2(x')}\left\{-\frac{\left(K_2(x',\xi')-1\right)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^2\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\right\}dx'-$$

$$-\sum_{j=1}^{2}\int_{\Gamma_2}\frac{\partial u_2(x)}{\partial x_j}\Big|_{x_3=\gamma_2(x')}\left(\left[\frac{(x_j-\xi_j)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^3\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\right]\cos(\nu_x,x_i)-\right.$$

$$\left.-\left[\frac{(x_i-\xi_i)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^3\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\right]\cos(\nu_x,x_j)dx\right)\frac{dx'}{\cos(\nu_x,x_3)}-$$

$$-\sum_{j=1}^{2}\int_{S}\frac{\partial u_2(x)}{\partial x_i}\Big|_{x_3=\gamma_2(x')}\left[\frac{(x_j-\xi_j)\theta(|x'-\xi'|\left(K(x',\xi')-1\right))}{2\pi|x'-\xi'|^3\sqrt{\left(K^2(x',\xi')-1\right)^3}}\right]\frac{\cos(\nu_x,x_j)dx'}{\cos(\nu_x,x_3)}+\dots,\ i=1,2,\quad (35)$$

and

$$\frac{1}{2}\frac{\partial u_2(\xi)}{\partial\xi_3}\Big|_{\xi_3=0}=\sum_{j=1}^{2}\int_{S}\frac{\partial u_2(x)}{\partial x_j}\Big|_{x_3=0}\left(-\frac{(x_j-\xi_j)\theta((-|x'-\xi'|)}{2\pi i|x'-\xi'|^3}\right)\cos(\nu_x,x_3)dx'+$$

$$+\sum_{j=1}^{2}\int_{S}\frac{\partial u_2(x)}{\partial x_3}\Big|_{x_3=0}\left[-\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|^3}\right]\cos(\nu_x,x_j)dx'+\dots,\quad (36)$$

and also

$$\frac{1}{2}\frac{\partial u_2(\xi)}{\partial\xi_3}\Big|_{\xi\in\Gamma_2}=\frac{1}{2}\frac{\partial u_2(\xi)}{\partial\xi_3}\Big|_{\xi_3=\gamma_2(\xi')}=$$

$$=\int_{S}\frac{\partial u_2(x)}{\partial x_3}\Big|_{x_3=\gamma_2(x')}\left\{-\frac{\left(K_2(x',\xi')-1\right)(\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^2\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\right\}dx'-$$

$$-\sum_{j=1}^{2}\int_{S}\frac{\partial u_2(x)}{\partial x_j}\Big|_{x_3=\gamma_2(x')}\left\{\frac{(x_j-\xi_j)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^3\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\cos(\nu_x,x_3)+\right.$$

$$\left.+\frac{\left(K_2(x',\xi')-1\right)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^2\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\cos(\nu_x,x_j)\right\}\frac{dx'}{\cos(\nu_x,x_3)}-$$

$$-\sum_{j=1}^{2}\int_{S}\frac{\partial u_2(x)}{\partial x_3}\Big|_{x_3=\gamma_2(x')}\frac{(x_j-\xi_j)\theta(|x'-\xi'|\left(K_2(x',\xi')-1\right))}{2\pi|x'-\xi'|^3\sqrt{\left(K_2^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_j)dx'}{\cos(\nu_x,x_3)}.\quad (37)$$

Theorem 5. *Under the conditions of Theorem* 4, *the necessary conditions* (32)–(37) *are singular*.

5. Regularization.

We will construct linear combinations from the boundary values of the desired functions and their partial derivatives $u_2(\xi)\big|_{\xi_3=\gamma_k(\xi')}, k=0,2, \frac{\partial u_1(\xi)}{\partial \xi_j}\big|_{\xi_3=\gamma_k(\xi')}, k=0,1,$ $\frac{\partial u_2(\xi)}{\partial \xi_j}\big|_{\xi_3=\gamma_k(\xi')}, k=0,2; j=\overline{1,3}, \quad \xi'\in S$, using coefficients $\beta_{i2}^{(k)}(\xi')$, $k=0,2$, $\beta_{ij1}^{(k)}(\xi')$, $k=0,1$, $\beta_{ij2}^{(k)}(\xi'), k=0,2$, $j=\overline{1,3}$; $i=1,2$, $\xi'\in S$:

$$\sum_{\substack{k=0,\\k\neq 1}}^{2}\beta_{i2}^{(k)}(\xi')u_2(\xi)\Big|_{\xi_3=\gamma_k(\xi')}+\sum_{j=1}^{3}\sum_{k=0}^{1}\beta_{ij1}^{(k)}(\xi')\frac{\partial u_1(\xi)}{\partial \xi_j}\Big|_{\xi_3=\gamma_k(\xi')}+$$
$$+\sum_{j=1}^{3}\sum_{\substack{k=0,\\k\neq 1}}^{2}\beta_{ij2}^{(k)}(\xi')\frac{\partial u_2(\xi)}{\partial \xi_j}\Big|_{\xi_3=\gamma_k(\xi')},\ i=1,2,\ \xi'\in S, \tag{38}$$

and then we substitute the singular necessary conditions (21) и (32)-(37) в (38).

Adding and subtracting $\beta_{ij}^{(k)}(x')$ from $\beta_{ij}^{(k)}(\xi')$ and $\beta_{ijl}^{(k)}(x')$ from $\beta_{ijl}^{(k)}(\xi')$ and, assuming that the above functions $\beta_{ji}^{(k)}(\xi'), \beta_{ijl}^{(k)}(\xi')$ satisfy the Hölder condition, we obtain weak singularities in the integrals with $\frac{\beta_{ji}^{(k)}(\xi')-\beta_{ji}^{(k)}(x')}{2\pi\left|x'-\xi'\right|^2}$. By discarding terms with weak singularities and expanding the coefficients at the boundary values $u_2(x)$, and partial derivatives $\frac{\partial u_k(x)}{\partial x_j}, j=\overline{1,3}; k=1,2$, with respect to x' at the point $x'=\xi'$ using the Taylor formula, we isolate only the first terms of the expansion.

By grouping the terms under the integral sign and equating the coefficients at $u_2(x)\big|_{x_3=\gamma_k(x')}, k=0,2,$ $\frac{\partial u_1(x)}{\partial x_j}\big|_{x_3=\gamma_k(x')}, k=0,1,$ $\frac{\partial u_2(x)}{\partial x_j}\big|_{x_3=\gamma_k(x')}, k=0,2; j=\overline{1,3}$, to the corresponding coefficients in the boundary conditions (2), we obtain a system of 14 equations for 14 unknowns

$\beta_{i2}^{(k)}(x')$, $k=0,2$, $\beta_{ij1}^{(k)}(x')$, $k=0,1$, $\beta_{ij2}^{(k)}(x')$, $k=0,2$, $j=\overline{1,3}$; $i=1,2$, $x'\in S$:

$$\beta_{i2}^{(0)}(x')\left(-\sum_{j=1}^{2}\frac{(x_j-\xi_j)\theta((-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_j)\right)_{\xi'=x'}=\alpha_{i2}^{(0)}(x'),\tag{39}$$

$$\beta_{i2}^{(2)}(x')\left(-\sum_{j=1}^{2}\frac{(x_j-\xi_j)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{(K^2(x',\xi')-1)^3}}\frac{\cos(\nu_x,x_j)}{\cos(\nu_x,x_3)}\right)\Bigg|_{\xi'=x'}=\alpha_{i2}^{(2)}(x'),\tag{40}$$

$$\beta_{i31}^{(0)}(x')\left[-\frac{K_{31}^{(0)}(x',\xi')}{4\pi P_0(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}+\beta_{i21}^{(0)}(x')\left[\frac{K_{21}^{(0)}(x',\xi')}{4\pi P_0(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}=\alpha_{i11}^{(0)}(x'),\tag{41}$$

$$\beta_{i11}^{(0)}(x')\left[-\frac{K_{12}^{(0)}(x',\xi')}{4\pi P_0(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}+\beta_{i31}^{(0)}(x')\left[\frac{K_{32}^{(0)}(x',\xi')}{4\pi P_0(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}=\alpha_{i21}^{(0)}(x'),\tag{42}$$

$$\beta_{i21}^{(0)}(x')\left[-\frac{K_{23}^{(0)}(x',\xi')}{4\pi P_0(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}+\beta_{i11}^{(0)}(x')\left[\frac{K_{13}^{(0)}(x',\xi')}{4\pi P_0(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}=\alpha_{i31}^{(0)}(x'),\tag{43}$$

$$\beta_{i31}^{(1)}(x')\left[-\frac{K_{31}^{(1)}(x',\xi')}{4\pi P_1(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}+\beta_{i21}^{(1)}(x')\left[\frac{K_{21}^{(1)}(x',\xi')}{4\pi P_1(x',\xi')\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}=\alpha_{i11}^{(1)}(x'),\tag{44}$$

$$\beta_{i11}^{(1)}(x')\left[-\frac{K_{12}^{(1)}(x',\xi')}{4\pi P_1(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}+\beta_{i31}^{(1)}(x')\left[\frac{K_{32}^{(1)}(x',\xi')}{4\pi P_1(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Bigg|_{\xi'=x'}=\alpha_{i21}^{(1)}(x'),\tag{45}$$

$$\beta^{(1)}_{i11}(x')\left[\frac{K^{(1)}_{13}(x',\xi')}{4\pi P_1(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+$$

$$\beta^{(1)}_{i21}(x')\left[-\frac{K^{(1)}_{23}(x',\xi')}{4\pi P_1(x',\xi')}\frac{1}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}=\alpha^{(1)}_{i31}(x'),\ (46)$$

$$\beta^{(0)}_{i32}(x')\left[-\frac{(x_1-\xi_1)\theta((-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_0)\right]\Big|_{\xi'=x'}+$$

$$+\sum_{j=1}^{2}\beta^{(0)}_{ij2}(x')\left[\left(\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_1)-\left(\frac{(x_1-\xi_1)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_j)\right]\Big|_{\xi'=}$$

, (47)

$$\sum_{j=1}^{2}\beta^{(0)}_{ij2}(x')\left[\left(\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_2)-\left(\frac{(x_2-\xi_2)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_j)\right]\Big|_{\xi'=x}$$

$$+\beta^{(0)}_{i32}(x')\left[-\frac{(x_2-\xi_2)\theta((-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_3)\right]\Big|_{\xi'=x'}=\alpha^{(0)}_{i22}(x'),$$

(48)

$$\sum_{j=1}^{2}\beta^{(0)}_{ij2}(x')\left[-\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\cos(\nu_x,x_3)\right]\Big|_{\xi'=x'}+$$

$$+\beta^{(0)}_{i32}(x')\sum_{j=1}^{2}\left[-\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_j)\right]\Big|_{\xi'=x'}=\alpha^{(0)}_{i32}(x'),$$

(49)

$$\beta^{(2)}_{i32}(x')\left[\frac{(x_1-\xi_1)\theta(|x'-\xi'|(K(x',\xi')-1))}{\pi|x'-\xi'|\sqrt{(K^2(x',\xi')-1)^3}}+\frac{K(x',\xi')\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi\sqrt{(K^2(x',\xi')-1)^3}}\frac{\cos(\nu_x,x_1}{\cos(\nu_x,x_3}\right.$$

$+$

$$+\beta^{(2)}_{i12}(x')\left\{\sum_{k=1}^{2}\left[-\frac{(x_k-\xi_k)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{(K^2(x',\xi')-1)^3}}\frac{\cos(\nu_x,x_k)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+\right.$$

$$+\beta_{i22}^{(2)}(x')\left[-\frac{(x_2-\xi_2)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_1)}{\cos(\nu_x,x_3)}+\right.$$

$$\left.+\frac{(x_1-\xi_1)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_2)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}=\alpha_{i12}^{(2)}(x'),\tag{50}$$

$$\beta_{i12}^{(2)}(x')\left[-\frac{(x_2-\xi_2)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_1)}{\cos(\nu_x,x_3)}\right.$$

$$\left.+\frac{(x_1-\xi_1)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_2)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+$$

$$+\beta_{i22}^{(2)}(x')\sum_{k=1}^{2}\left[-\frac{(x_k-\xi_k)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_k)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+$$

$$+\left[-\frac{(x_2-\xi_2)\theta(|x'-\xi'|(K(x',\xi')-1))}{\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}-\frac{K(x',\xi')\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_2)}{\cos(\nu_x,x_3)}\right]\Big|$$

$$=\alpha_{i22}^{(2)}(x'),\tag{51}$$

$$\beta_{i32}^{(2)}(x')\left\{\sum_{j=1}^{2}\left[-\frac{(x_j-\xi_j)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_j)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+\right.$$

$$+\sum_{k=1}^{2}\left\{\left[-\beta_{ik2}^{(2)}(x')\frac{K(x',\xi')\theta(|x'-\xi'|(K(x',\xi')-1))}{\pi\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_k)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+\right.$$

$$+\left[-\beta_{ik2}^{(2)}(x')\frac{(x_k-\xi_k)\theta(|x'-\xi'|(K(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}\right]\Bigg|_{\xi'=x'}\Bigg\}=\alpha_{i32}^{(2)}(x'). \tag{52}$$

Let us assume that

$$\left(-\sum_{j=1}^{2}\frac{(x_j-\xi_j)\theta((-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_j)\right)\Bigg|_{\xi'=x'}\neq 0 \ , \tag{53}$$

$$\left(-\sum_{j=1}^{2}\frac{(x_j-\xi_j)\theta(|x'-\xi'|(K_2(x',\xi')-1))}{2\pi|x'-\xi'|\sqrt{\left(K_2{}^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_j)}{\cos(\nu_x,x_3)}\right)\Bigg|_{\xi'=x'}\neq 0 \ . \tag{54}$$

Then to find the coefficients $\beta_{i31}^{(k)}(x')$, $\beta_{i21}^{(k)}(x')$, $\beta_{i11}^{(k)}(x'), k=0,1$, introducing the notations

$$\Delta_{s1}^{(k)}(x')=(-1)^s\frac{K_{3s}^{(k)}(x',x')}{4\pi P_k(x',x')}\frac{1}{\cos(\nu_x,x_3)}, s=1,2;\ \Delta_{12}^{(k)}(x')=$$

$$\frac{K_{21}^{(k)}(x',x')}{4\pi P_k(x',x')}\frac{1}{\cos(\nu_x,x_3)},\quad \Delta_{sl}^{(k)}(x')=0, s+l=4, s>0, l>0,$$

$$\Delta_{23}^{(k)}(x')=\frac{K_{12}^{(k)}(x',x')}{4\pi P_k(x',x')}\frac{1}{\cos(\nu_x,x_3)},$$

$$\Delta_{32}^{(k)}(x')=-\frac{K_{23}^{(k)}(x',x')}{4\pi P_k(x',x')}\frac{1}{\cos(\nu_x,x_3)},\quad \Delta_{33}^{(k)}(x')=\frac{K_{13}^{(k)}(x',x')}{4\pi P_k(x',x')}\frac{1}{\cos(\nu_x,x_3)},$$

we assume that

$$\Delta^{(k)}(x')=\begin{vmatrix}\Delta_{11}^{(k)}(x') & \Delta_{12}^{(k)}(x') & \Delta_{13}^{(k)}(x')\\ \Delta_{21}^{(k)}(x') & \Delta_{22}^{(k)}(x') & \Delta_{23}^{(k)}(x')\\ \Delta_{31}^{(k)}(x') & \Delta_{32}^{(k)}(x') & \Delta_{33}^{(k)}(x')\end{vmatrix}\neq 0, k=0,1. \tag{55}$$

For the coefficients at the unknowns $\beta_{i32}^{(0)}(x')$, $\beta_{i22}^{(0)}(x')$, $\beta_{i12}^{(0)}(x')$

$$\Delta_{11}^{(3)}(x')=\left[-\frac{(x_1-\xi_1)\theta((-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_0)\right]\Bigg|_{\xi'=x'},$$

$$\Delta_{12}^{(3)}(x') =$$
$$\left[\left(\frac{(x_2-\xi_2)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_1)-\left(\frac{(x_1-\xi_1)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_2)\right]\Big|_{\xi'=x'},$$

$$\Delta_{13}^{(3)}(x')=0,\ \Delta_{22}^{(3)}(x')=0,$$

$$\Delta_{21}^{(3)}(x')=\left[-\frac{(x_2-\xi_2)\theta((-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_0)\right]\Big|_{\xi'=x'},$$

$$\Delta_{23}^{(3)}(x') =$$
$$\left[\left(\frac{(x_1-\xi_1)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_2)-\left(\frac{(x_2-\xi_2)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\right)\cos(\nu_x,x_1)\right]\Big|_{\xi'=x'}$$

$$\Delta_{31}^{(3)}(x')=\sum_{j=1}^{2}\left[-\frac{(x_j-\xi_j)\theta(-|x'-\xi'|)}{\pi i|x'-\xi'|}\cos(\nu_x,x_j)\right]\Big|_{\xi'=x'},$$

$$\Delta_{32}^{(3)}(x')=\left[-\frac{(x_2-\xi_2)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\cos(\nu_x,x_3)\right]\Big|_{\xi'=x'},$$

$$\Delta_{33}^{(3)}(x')=\left[-\frac{(x_1-\xi_1)\theta(-|x'-\xi'|)}{2\pi i|x'-\xi'|}\cos(\nu_x,x_3)\right]\Big|_{\xi'=x'},$$

we assume that

$$\Delta^{(3)}(x')=\begin{vmatrix}\Delta_{11}^{(3)}(x') & \Delta_{12}^{(3)}(x') & \Delta_{13}^{(3)}(x')\\ \Delta_{21}^{(3)}(x') & \Delta_{22}^{(3)}(x') & \Delta_{23}^{(3)}(x')\\ \Delta_{31}^{(3)}(x') & \Delta_{32}^{(3)}(x') & \Delta_{33}^{(3)}(x')\end{vmatrix}\neq 0. \tag{56}$$

For the coefficients at the unknowns $\beta_{i32}^{(2)}(x')$, $\beta_{i22}^{(2)}(x')$, $\beta_{i12}^{(2)}(x')$

$$\Delta_{11}^{(4)}(x') =$$
$$=\left[\frac{(x_1-\xi_1)\theta(|x'-\xi'|\left(K(x',\xi')-1\right))}{\pi|x'-\xi'|\sqrt{\left(K^2(x',\xi')-1\right)^3}}+\frac{K(x',\xi')\theta(|x'-\xi'|\left(K(x',\xi')-1\right))}{2\pi\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_1)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}$$
,

$$\Delta_{12}^{(4)}(x')=0,$$

$$\Delta_{13}^{(4)}(x') = \frac{1}{4\pi}\times$$

$$\left\{\sum_{k=1}^{2}\left[-\frac{\cos(x'-\xi,x_k)\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_k)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+\left[-\frac{K(x',x')\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}\right]\Big|_{\xi'=x'}\right.$$

$$\left[\left(-\frac{\cos(x'-\xi',x_2)\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_1)}{\cos(\nu_x,x_3)}+\frac{\cos(x'-\xi',x_1)\left(K(x',\xi')-1\right)}{\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_2)}{\cos(\nu_x,x_3)}\right)\right.$$

$$\Delta_{21}^{(3)}(x') =$$

$$\frac{1}{2\pi}\left[-\frac{\cos(x'-\xi',x_2)\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}-\frac{K(x',x')\left(K(x',\xi')-1\right)}{2\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_2)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}$$

,

$$\Delta_{22}^{(4)}(x') = \sum_{k=1}^{2}\left[-\frac{\cos(x'-\xi,x_k)\left(K(x',x')-1\right)}{4\pi\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_k)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'},$$

$$\Delta_{23}^{(4)}(x') =$$

$$\frac{1}{4\pi}\left[-\frac{\cos(x'-\xi',x_2)\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_1)}{\cos(\nu_x,x_3)}+\frac{\cos(x'-\xi',x_1)\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_2)}{\cos(\nu_x,x_3)}\right.$$

,

$$\Delta_{31}^{(4)}(x') = -\frac{1}{2\pi}\left\{\sum_{j=1}^{2}\left[\frac{\cos(x'-\xi',x_j)\left(K(x',x')-1\right)}{2\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_j)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+\right.$$

$$\left.\frac{\left(K(x',x')-1\right)^2}{\sqrt{\left(K^2(x',\xi')-1\right)^3}}\frac{\cos(\nu_x,x_3)}{\cos(\nu_x,x_3)}\right\},$$

$$\Delta_{32}^{(4)}(x') =$$
$$-\frac{1}{2\pi}\left\{\left[\frac{K(x',x')\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_2)}{\cos(\nu_x,x_3)}\right]\Big|_{\xi'=x'}+\left[\frac{\cos(x'-\xi',x_2)\left(K(x',x')-1\right)}{2\sqrt{\left(K^2(x',x')-1\right)^3}}\right]\Big|_{\xi'=x'}\right\}$$
,

$$\Delta_{33}^{(4)}(x') =$$
$$-\frac{1}{2\pi}\left\{\frac{K(x',x')\left(K(x',x')-1\right)}{\sqrt{\left(K^2(x',x')-1\right)^3}}\frac{\cos(\nu_x,x_1)}{\cos(\nu_x,x_3)}+\frac{\cos(x'-\xi',x_1)\left(K(x',x')-1\right)}{2\sqrt{\left(K^2(x',x')-1\right)^3}}\right\}\Big|_{\xi'=x'}$$
,

we assume that

$$\Delta^{(4)}(x') = \begin{vmatrix} \Delta_{11}^{(4)}(x') & \Delta_{12}^{(4)}(x') & \Delta_{13}^{(4)}(x') \\ \Delta_{21}^{(4)}(x') & \Delta_{22}^{(4)}(x') & \Delta_{23}^{(4)}(x') \\ \Delta_{31}^{(4)}(x') & \Delta_{32}^{(4)}(x') & \Delta_{33}^{(4)}(x') \end{vmatrix} \neq 0 . \tag{57}$$

Solving the system (39)-(52) we obtain the linear combinations (38) in the form:

$$\sum_{k=0;2}\beta_{i2}^{(k)}(\xi')u_2(\xi)\Big|_{\xi_3=\gamma_k(\xi')}+\sum_{j=1}^{3}\sum_{k=0}^{1}\beta_{ij1}^{(k)}(\xi')\frac{\partial u_1(\xi)}{\partial\xi_j}\Big|_{\xi_3=\gamma_k(\xi')}+$$
$$\sum_{j=1}^{3}\sum_{\substack{k=0,\\k\neq1}}^{2}\beta_{ij2}^{(k)}(\xi')\frac{\partial u_2(\xi)}{\partial\xi_j}\Big|_{\xi_3=\gamma_k(\xi')}=$$
$$=\int_S\frac{f_i(x')}{|x'-\xi'|^2}dx'-\int_S\frac{1}{|x'-\xi'|^2}\sum_{m=0}^{1}\alpha_{i1}^{(m)}(x')u_1(x',\gamma_m(x'))dx'+\dots,\ i=1,2 . \tag{58}$$

The first integral on the right-hand side in (58) converges, since by the condition of the functions $f_i(x')\in C^1(S),\ f_i(x')\big|_{\partial S}=0,\ i=1,2$. Substituting the regular necessary conditions (19) into (58) and changing the order of integration, we obtain the inner integrals that do not depend on the desired function and its derivatives and converge in the Cauchy sense.

Thus, we have regularized the necessary conditions (21), (32)-(37) and arrived at the following statement.

Theorem 6. Let $D\subset R^3$ be a bounded convex domain in direction Ox_3 with boundary $\Gamma=\partial D=\Gamma_1\cup\Gamma_2$ being a Lyapunov surface with equations $x_3=\gamma_k(x')\in C^2(S),\ S=\underset{x_3=0}{pr}\,D, k=1,2$, boundary conditions (2) being linearly independent; the coefficients $\alpha_{ijp}^{(k)}(x'),\alpha_{ip}^{(k)}(x')$, $i,p=1,2;\quad j=1,2,3;\ k=0,1$, be continuous in S and $\alpha_{i2}^{(k)}(\xi')$, $k=0,2$, $\alpha_{ij1}^{(k)}(\xi')$, $k=0,1$, $\alpha_{ij2}^{(k)}(\xi')$, $k=0,2;\ j=\overline{1,3},\ i=1,2;\ \xi'\in S$ in (2) belong to some Hölder class, the right-hand sides $f_i(x')\in C^1(S),\quad f_i(x')\big|_{\partial S}=0,\ i=1,2$, $f_0(x')\in C(L)$. Then, if conditions (53)-(57) are satisfied, the relationships (58), $i=1,2$, are regular.

5. Conclusion.

A boundary value problem for a three-dimensional mixed-type equation with nonlocal boundary conditions in a bounded domain with a Lyapunov boundary is investigated. Necessary solvability conditions are derived and regularized in both the elliptic and hyperbolic cases using an original scheme.

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